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A momentum subtraction scheme for two–nucleon effective field theory

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Abstract

We introduce a momentum subtraction scheme which obeys the power counting of Kaplan, Savage, and Wise (KSW), developed for systems with large scattering lengths, a. Unlike the power divergence subtraction scheme, coupling constants in this scheme obey the KSW scaling for all $\mu_R > 1/a$. We comment on the low-energy theorems derived by Cohen and Hansen. We conclude that there is no obstruction to using perturbative pions for momenta $p > m_\pi$.

Effective field theory is a useful method for describing two-nucleon systems [1]. Recently, Kaplan, Savage, and Wise (KSW) [2] devised a power counting that accounts for the effect of large scattering lengths. With this power counting the dimension six four-nucleon operators are treated non-perturbatively, while pion exchange as well as higher dimension operators are treated perturbatively. A key feature of the KSW approach is the use of the power divergence subtraction (PDS) renormalization scheme. PDS gives the coefficients of contact interactions power law dependence on the renormalization scale, which makes the KSW power counting manifest. For $\mu_R \gtrsim 300\,\mathrm{MeV}$, the power counting in PDS is no longer manifest [2], and KSW conclude that application of the theory is restricted to momenta less than 300 MeV.

In Ref. [3], it is emphasized that the power counting for large scattering lengths should be scheme independent. Here we introduce a momentum subtraction scheme that is compatible with the KSW power counting. A similar scheme is applied to the theory without pions in Ref. [4]. This scheme will be referred to as the OS scheme, since in a relativistic field theory it would be called an off-shell momentum subtraction scheme. One attractive feature of the OS scheme is that the KSW power counting is manifest for all $\mu_R > 1/a$, where a is the scattering length. Since the breakdown of the power counting is an artifact of the PDS scheme, the range of the effective field theory with perturbative pions may not be limited to 300 MeV.

Another attractive feature of the OS scheme is that the amplitudes are manifestly μ_R independent at each order in the expansion. In PDS, any fixed order calculation has residual μ_R dependence which is cancelled by higher order terms. The rather strong μ_R dependence exhibited by next-to-leading order calculations of the phase shifts led authors [5,6] to conclude that PDS results have to be fine tuned to fit the data. In fact, it is possible to fix this problem with PDS by modifying the treatment of $C_0(\mu_R)$.

We begin our discussion by recalling the Lagrangian with pions and nucleons [2]

$$\begin{aligned}\mathcal{L}_\pi = {} & \frac{f^2}{8}\mathrm{Tr}\left(\partial^\mu\Sigma\,\partial_\mu\Sigma^\dagger\right) + \frac{f^2\omega}{4}\,\mathrm{Tr}(m_q\Sigma + m_q\Sigma^\dagger) - \frac{ig_A}{2}\,N^\dagger\sigma_i(\xi\partial_i\xi^\dagger - \xi^\dagger\partial_i\xi)N \\ & + N^\dagger\Big(iD_0 + \frac{\vec{D}^2}{2M}\Big)N - C_0^{(s)}(N^T P_i^{(s)} N)^\dagger(N^T P_i^{(s)} N) \\ & - \frac{C_2^{(s)}}{8}\left[(N^T P_i^{(s)} N)^\dagger(N^T P_i^{(s)} \overleftrightarrow{\nabla}^2 N) + h.c.\right] - D_2^{(s)}\omega\mathrm{Tr}(m^\xi)(N^T P_i^{(s)} N)^\dagger(N^T P_i^{(s)} N) + \ldots .\end{aligned} \tag{1}$$

Here $g_A = 1.25$ is the nucleon axial-vector coupling, $f = 130\,\mathrm{MeV}$ is the pion decay constant, $m^\xi = \frac{1}{2}(\xi m_q\xi + \xi^\dagger m_q\xi^\dagger)$, where $m_q = \mathrm{diag}(m_u, m_d)$ is the quark mass matrix, and $m_\pi^2 =$

$w(m_u + m_d)$. The matrices $P_i^{(s)}$ project onto states of definite spin and isospin, and the superscript s denotes the partial wave amplitude mediated by the operator. This paper will be concerned only with S-wave scattering, so $s = {}^1S_0, {}^3S_1$. The ellipsis in Eq. (1) denotes terms that are higher order in the expansion.

The four nucleon couplings C_0, C_2, and D_2 in Eq. (1) are bare parameters. In general, a bare coupling, $C^{\rm bare}$, can be separated into a renormalized coupling, $C(\mu_R)$, and counterterms:

$$C^{\rm bare} = C^{\rm finite} - \delta^{\rm uv} C\,, \qquad C^{\rm finite} = C(\mu_R) - \sum_{n=0}^{\infty} \delta^n C(\mu_R)\,. \tag{2}$$

The counterterms have been divided into two classes. The first, which have the superscript uv, contain all ultraviolet divergences and are μ_R independent. Defining a renormalization scheme amounts to making a choice for the finite counterterms, $\delta^n C(\mu_R)$. The superscript n indicates that $\delta^n C$ is included at tree level for a graph with n loops. For example, at two loops, we have two loop diagrams with renormalized couplings at the vertices, one loop diagrams with a single $\delta^1 C$ counterterm, and a tree level diagram with $\delta^2 C$. Let Q denote a typical momentum characterizing the process under consideration. For nucleon-nucleon scattering we take $p \sim Q$ and $m_\pi \sim Q$, where p is the center-of-mass momentum. The theory is an expansion in Q/Λ where Λ is the range of the effective field theory. Taking $\mu_R \sim Q$, vertices with $C_0(\mu_R)$ scale as $1/Q$, while vertices with $C_2 p^2$ or $D_2 m_\pi^2$ scale as Q^0. A typical loop gives one power of Q, so $C_0(\mu_R)$ vertices are included to all orders. This sums all corrections that scale as $(Qa)^n$ [2]. Note that since the pion has been included explicitly in the Lagrangian we expect that the scale of short distance physics, Λ, should not be set by m_π, but by higher mass resonances which have not been included in the theory.

PDS is one scheme in which the KSW power counting is manifest. In PDS, we first let $d = 4-2\epsilon$ and define the counterterms $\delta^{\rm uv} C$ to subtract $1/\epsilon$ poles. As in the $\overline{\rm MS}$ scheme, the dimensional regularization parameter μ is set to μ_R. Next one takes $d = 3$ and defines the finite counterterms, $\delta^n C(\mu_R)$ to subtract the $1/(d-3)$ poles in the amplitude. Graphs which renormalize a given coupling are those whose vertices have the right number of derivatives and powers of m_π^2. When calculating $\delta^n C_0(\mu_R)$ and $\delta^n C_2(\mu_R)$, we can take $m_\pi = 0$ since counterterms proportional to m_π^2 renormalize coefficients like $D_2(\mu_R)$. After making these subtractions, the amplitude is continued back to four dimensions, so $d - 3 \to 1$.

In the ${}^1S_0$ channel, exact expressions for the PDS beta functions can be obtained [2]. For $C_0(\mu_R)$ we have

$$C_0(\mu_R) = \frac{4\pi}{M}\left(\frac{1}{K-\mu_R} - \frac{1}{\Lambda_{NN}}\right), \tag{3}$$

where $\Lambda_{NN} = 8\pi f^2/(Mg_A^2) = 300\,\mathrm{MeV}$. K is a constant fixed by the boundary condition, and choosing $C_0(0) = 4\pi a/M$ gives $K = 1/(a+1/\Lambda_{NN})$. We see that the scaling for $C_0(\mu_R)$ changes for $\mu_R \sim 300\,\mathrm{MeV}$ [2]. A simple shift, $C_0(\mu_R) \to C_0(\mu_R) - g_A^2/2f^2$, results in a coupling that scales as $1/\mu_R$ for all $\mu_R > 1/a$. Physically, this shift corresponds to summing the short distance ($m_\pi = 0$) contributions from potential pion exchange to all orders. For the ${}^3S_1$ channel, this summation is not possible because of ultraviolet divergences of the form p^2/ϵ and, in fact, there are unknown corrections to the ${}^3S_1$ beta functions at each order in Q. This is demonstrated in Ref. [7] by explicit computation of the counterterms. When the beta function is not exactly known, the large μ_R behavior is ambiguous. For example, the PDS beta function for $C_0(\mu_R)$ is

$$\beta_0 = \mu_R \frac{\partial C_0(\mu_R)}{\partial \mu_R} = \frac{M\mu_R}{4\pi}\left[C_0^2(\mu_R) + 2\frac{g_A^2}{2f^2}C_0(\mu_R)\right] + \mathcal{O}(Q)\,. \tag{4}$$

Two solutions which satisfy this equation to order Q^0 are

$$\begin{aligned} C_0(\mu_R) &= -\frac{4\pi}{M}\frac{\left[1-2a\mu_R-2\mu_g-\sqrt{1+4\mu_g^2-4\mu_g(1-a\mu_R)}\,\right]}{2\mu_R(1-a\mu_R-\mu_g)}\,, \\ C_0(\mu_R) &= -\frac{g_A^2}{f^2}\frac{1}{1-\left[1+2/(a\Lambda_{NN})\right]\exp\left(-2\mu_g\right)}\,, \end{aligned} \tag{5}$$

where $\mu_g = \mu_R/\Lambda_{NN}$ and we have chosen $C_0(0) = 4\pi a/M$. The first solution is obtained by computing the counterterms $\delta^n C_0(\mu_R)$ to order Q^0 and summing them. This solution falls as $1/\mu_R$ for all $\mu_R > 1/a$, and is numerically close to the $g_A \to 0$ solution. The second solution is obtained by truncating and solving Eq. (4). This solution approaches a constant as $\mu_R \to \infty$. The two solutions both solve the beta function to order Q^0 but have very different large μ_R behavior. In the OS scheme, there is no ambiguity since at a given order in Q the running of all the coupling constants that enter is known exactly.

In the OS momentum subtraction scheme, the renormalized couplings are defined by relating them to the amplitude evaluated at the unphysical momentum $p = i\mu_R$. We start by dividing up the full amplitude as $A = \sum_{m=0}^{\infty} A^{(m-1)}$. Here $A^{(m-1)}$ contains the Feynman diagrams that scale as Q^{m-1}. As in PDS, we can take $m_\pi \to 0$ when computing the counterterms for $C_0(\mu_R)$ and $C_2(\mu_R)$. $\delta^{\mathrm{uv}} C_{2m}$ is first defined to subtract all four dimensional $1/\epsilon$ poles. The definition for the renormalized coupling is then

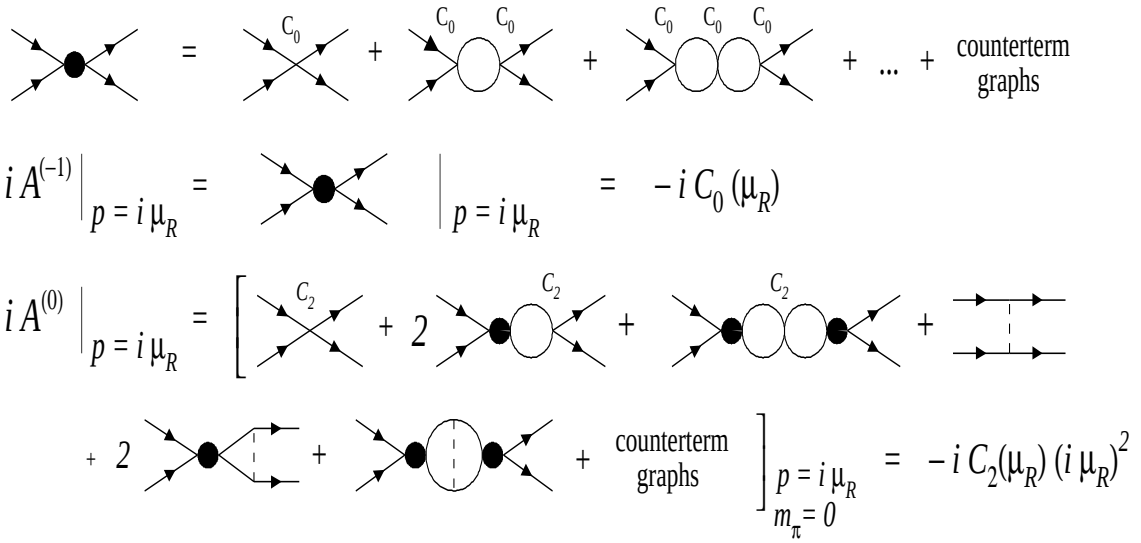

FIG. 1. Renormalization conditions for $C_0(\mu_R)$ and $C_2(\mu_R)$ in the OS scheme. $i\,A^{(-1)}$ is the four point function with $C_0(\mu_R)$ and $\delta^n C_0(\mu_R)$ vertices, evaluated between incoming and outgoing ${}^1S_0$ or ${}^3S_1$ states. The amplitude $A^{(0)}$ contains graphs with one C_2 or one potential pion dressed with C_0 bubbles.

$$iA^{(m-1)}\Big|_{\substack{p\,=\,i\mu_R\\ m_\pi\,=\,0}} = -iC_{2m}(\mu_R)\,(i\mu_R)^{2m}\,. \tag{6}$$

This condition is to be imposed order by order in the loop expansion so that the graphs at n loops determine $\delta^n C_0(\mu_R)$. For instance, the couplings $C_0(\mu_R)$ and $C_2(\mu_R)$ are defined by the renormalization condition in Fig. 1. Summing the counterterms, we find

$$C_0(\mu_R) = \frac{C_0^{\rm finite}}{1-\mu_R\frac{C_0^{\rm finite}M}{(4\pi)}}\,, \qquad C_2(\mu_R) = \frac{C_2^{\rm finite}-g_A^2/(2f^2\mu_R^2)}{\left[1-\mu_R\frac{C_0^{\rm finite}M}{4\pi}\right]^2}\,. \tag{7}$$

Although it may seem that the piece of $C_2(\mu_R)$ that goes as $1/\mu_R^4$ will spoil the power counting for low momentum, in fact, the $1/\mu_R^2$ part dominates entirely for $\mu_R \gtrsim 1/a$, since $C_0^{\rm finite}\sim a$, $C_2^{\rm finite}\sim a^2$.

In the OS scheme, $D_2(\mu_R)$ will be calculated as follows. First m_π^2/ϵ poles are subtracted. The renormalized coupling is then defined by

$$i\,A(D_2)\Big|_{p=i\mu_R,} = -i\,D_2(\mu_R)\,m_\pi^2\,, \tag{8}$$

where $A(D_2)$ contains terms in the amplitude that are proportional to m_π^2, as well as analytic in m_π^2. Thus, at Q^0 graphs with a single $D_2(\mu_R)$ or potential pion and any number of $C_0(\mu_R)$

vertices contribute to $A(D_2)$. Note that we only keep terms that are analytic in m_π^2 because it seems unnatural to put long-distance nonanalytic contributions that come from pion exchange into the definition of the short distance coupling $D_2(\mu_R)$ [6]. For example, one potential pion exchange gives a $(m_\pi^2/p^2)\ln(1+4p^2/m_\pi^2)$ term. Putting this into $D_2(\mu_R)$ would give it both a branch cut at $\mu_R = m_\pi/2$ as well as explicit dependence on the scale m_π, therefore this term is not included in $A(D_2)$. Solving for the counterterms and summing we find

$$\frac{D_2(\mu_R)}{C_0(\mu_R)^2} = \frac{D_2^{\rm finite}}{(C_0^{\rm finite})^2} + \frac{M}{8\pi}\Big(\frac{Mg_A^2}{8\pi f^2}\Big)\left[\ln\Big(\frac{\mu_R^2}{\mu_0^2}\Big) - 1\right] = \frac{M}{8\pi}\Big(\frac{Mg_A^2}{8\pi f^2}\Big)\ln\Big(\frac{\mu_R^2}{\tilde\mu^2}\Big)\,,$$
$$\text{where } \tilde\mu^2 = \mu_0^2\,\exp\Big(1 - \frac{64\pi^2 f^2 D_2^{\rm finite}}{M^2 g_A^2 (C_0^{\rm finite})^2}\Big)\,. \tag{9}$$

With $m_\pi \sim Q \sim \mu_R$, $D_2(\mu_R)m_\pi^2 \sim Q^0$. The scale $\tilde\mu$ must be determined by fitting to data. The PDS solution for $D_2(\mu_R)$ does not have the -1 in square brackets.

We now compare the amplitudes in the PDS and OS schemes. At order Q^0, the amplitudes in the ${}^1S_0$ and ${}^3S_1$ channels have the same functional form. The amplitudes in PDS are calculated to this order in Ref. [2]. We find

$$A = A^{(-1)} + A^{(0,a)} + A^{(0,b)} + \mathcal{O}(Q^2)\,,$$
$$A^{(-1)} = -\frac{4\pi}{M}\,\frac{1}{\frac{4\pi}{MC_0(\mu_R)} + \mu_R + ip}\,, \tag{10}$$

$$\frac{A^{(0,a)}}{\Big[A^{(-1)}\Big]^2} = \frac{g_A^2 m_\pi^2}{2f^2}\Big(\frac{M}{4\pi}\Big)^2\left\{\frac{1}{2}\ln\Big(\frac{\mu_R^2}{m_\pi^2}\Big) - \Big(\frac{4\pi}{MC_0(\mu_R)} + \mu_R\Big)\frac{1}{p}\tan^{-1}\Big(\frac{2p}{m_\pi}\Big)\right. \tag{11}$$
$$\left. + \Big[\Big(\frac{4\pi}{MC_0(\mu_R)} + \mu_R\Big)^2 - p^2\Big]\frac{1}{4p^2}\ln\Big(1 + \frac{4p^2}{m_\pi^2}\Big)\right\} - \frac{D_2(\mu_R)\,m_\pi^2}{C_0(\mu_R)^2}\,,$$

PDS $$\frac{A^{(0,b)}}{\Big[A^{(-1)}\Big]^2} = -\frac{C_2(\mu_R)\,p^2}{C_0(\mu_R)^2} - \frac{g_A^2}{2f^2}\frac{1}{C_0(\mu_R)^2} + \frac{1}{2}\,\frac{g_A^2 m_\pi^2}{2f^2}\Big(\frac{M}{4\pi}\Big)^2\,, \tag{12}$$

OS $$\frac{A^{(0,b)}}{\Big[A^{(-1)}\Big]^2} = -\frac{C_2(\mu_R)\,p^2}{C_0(\mu_R)^2} - \frac{g_A^2}{2f^2}\Big(1 + \frac{p^2}{\mu_R^2}\Big)\Big(\frac{1}{C_0(\mu_R)} + \frac{M\mu_R}{4\pi}\Big)^2\,. \tag{13}$$

Note that the last term in Eq. (12) has a factor of 1/2 instead of a 1 as in [2] since we have made a different finite subtraction. The terms in braces are long distance pion contributions, and are the same in the PDS and OS schemes. By substituting Eqs. (7) and (9) into the OS amplitude one can verify that it is μ_R independent. In contrast, in PDS the amplitude is only μ_R independent to order Q^0.

To obtain a good fit to the scattering data at low momenta, two constraints must be approximately satisfied:

$$\lim_{p\to 0}\ p\cot\delta(p) = -\frac{1}{a}\,, \qquad\qquad \frac{A^{(0)}}{[A^{(-1)}]^2}\,\Big|_{-ip=\gamma} = 0\,, \tag{14}$$

where $\gamma = \frac{4\pi}{MC_0(\mu_R)} + \mu_R$. The second constraint ensures that the next-to-leading order amplitude does not have a spurious double pole. In the PDS scheme, $A^{(0)}$ is not μ_R independent, so the extracted parameters can not simultaneously satisfy the renormalization group equations and give a good fit. This is the origin of the large dependence on μ_R observed in Ref. [5]. In PDS the second constraint gives

$$\begin{aligned} 0 \simeq \frac{-D_2(\mu_R)}{C_0(\mu_R)^2} + \frac{g_A^2}{2f^2}\Big(\frac{M}{4\pi}\Big)^2\Big[\,\frac{1}{2} + \frac{1}{2}\ln\Big(\frac{\mu_R^2}{m_\pi^2}\Big) - \frac{2}{m_\pi}\Big(\frac{4\pi}{MC_0(\mu_R)} + \mu_R\Big) \\ + \frac{1}{m_\pi^2}\Big(\frac{4\pi}{MC_0(\mu_R)} + \mu_R\Big)^2 - \frac{1}{m_\pi^2}\Big(\frac{4\pi}{MC_0(\mu_R)}\Big)^2\,\Big]\,, \end{aligned} \tag{15}$$

while in OS we find

$$0 \simeq \frac{-D_2(\mu_R)}{C_0(\mu_R)^2} + \frac{g_A^2}{2f^2}\Big(\frac{M}{4\pi}\Big)^2\Big[\,\frac{1}{2}\ln\Big(\frac{\mu_R^2}{m_\pi^2}\Big) - \frac{2}{m_\pi}\Big(\frac{4\pi}{MC_0(\mu_R)} + \mu_R\Big)\Big]\,, \tag{16}$$

where terms of order γ^2/m_π^2 have been dropped. With the convention for $D_2(\mu_R)$ in Ref. [2] the first term in square brackets in Eq. (15) is a 1, so they find that $D_2(m_\pi)$ is close to zero. The last term in Eq. (15) induces the large μ_R dependence. On the other hand Eq. (16) is μ_R independent.

Fits are performed to the Nijmegen phase shift [8] data between 7 and 100 MeV for both the ${}^1S_0$ and ${}^3S_1$ channels. The phase shifts have an expansion of the form $\delta = \delta^{(0)} + \delta^{(1)} + \mathcal{O}(Q^2/\Lambda^2)$, where [2]

$$\delta^{(0)} = -\frac{i}{2}\ln\Big[1 + i\frac{pM}{2\pi}A^{(-1)}\Big]\,, \qquad \delta^{(1)} = \frac{pM}{4\pi}\frac{A^{(0)}}{1 + i\frac{pM}{2\pi}A^{(-1)}}\,. \tag{17}$$

The fits were weighted towards low momentum since the theoretical error is smallest there. The results are shown in Fig. 2. We chose to fit using the amplitudes in the OS scheme, but an equally good fit is obtained in PDS. The parameters $C_0(\mu_R)$, $C_2(\mu_R)$, and $D_2(\mu_R)$ were extracted for different values of μ_R in both schemes, and the conditions in Eq. (15) and (16) were found to be well satisfied. For instance, taking $\mu_R = m_\pi = 137\,\mathrm{MeV}$ in the OS scheme we find

$$\begin{aligned} {}^1S_0 \qquad & C_0(m_\pi) = -3.54\,\mathrm{fm}^2\,, \qquad C_2(m_\pi) = 3.00\,\mathrm{fm}^4\,, \qquad D_2(m_\pi) = 0.377\,\mathrm{fm}^4\,, \\ {}^3S_1 \qquad & C_0(m_\pi) = -5.81\,\mathrm{fm}^2\,, \qquad C_2(m_\pi) = 10.16\,\mathrm{fm}^4\,, \qquad D_2(m_\pi) = -4.128\,\mathrm{fm}^4\,. \end{aligned} \tag{18}$$

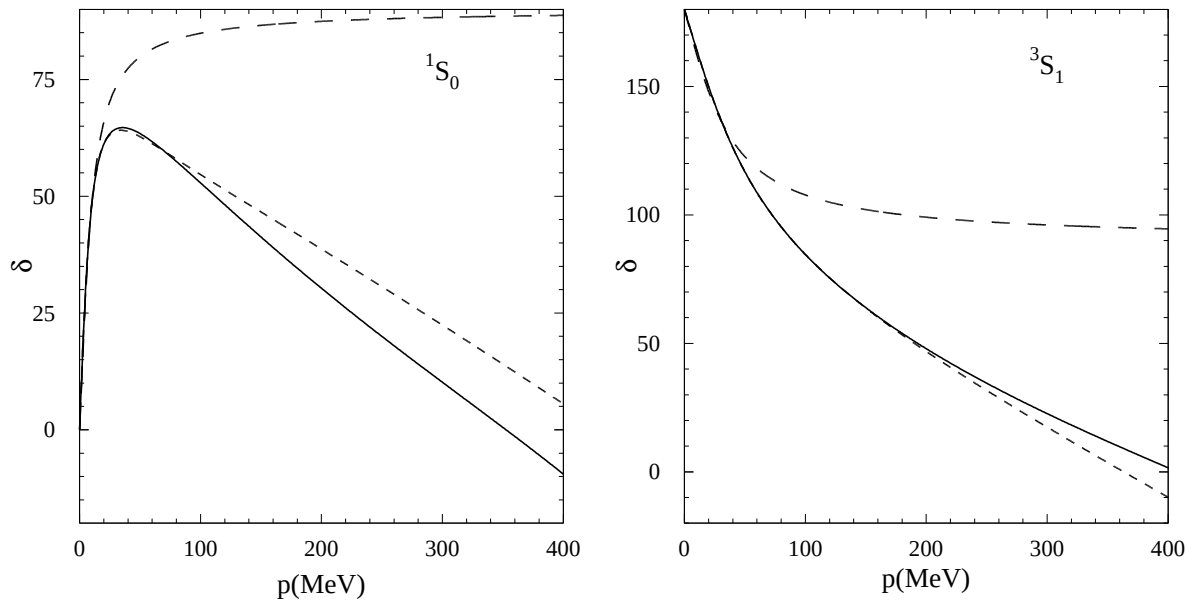

FIG. 2. Fit to the phase shift data weighted toward low momentum. The solid line is the Nijmegen fit to the data [8], the long dashed line is the order $1/Q$ result, and the short dashed line is the order Q^0 result.

The fit gives scattering lengths $a(^1S_0) = -23.3\,\mathrm{fm}$, and $a(^3S_1) = 5.41\,\mathrm{fm}$. Plugging the parameters extracted from the fit into the right hand side of Eq. (16) gives $\sim 10^{-2}$ for both channels. In the OS scheme, the renormalization group equations are obeyed to a few percent accuracy since the amplitude is explicitly μ_R independent.

It is important to realize that the fits do not unambiguously determine the values of $C_0(\mu_R)$ and $D_2(\mu_R)$. The coefficient of the four nucleon operator with no derivatives is $C_0^{bare}+m_\pi^2 D_2^{bare}$. Once we switch to renormalized coefficients, it is not clear how to divide the couplings into a nonperturbative and perturbative piece. In Ref. [2], $C_0(\mu)$ is summed to all orders, while other authors [5,6] treat both coefficients non-perturbatively. In fact, in order to do a chiral expansion, $m_\pi^2 D_2(\mu_R)$ should be treated perturbatively. Since $m_\pi^2 D_2(\mu_R) \sim Q^0$, this is consistent with the power counting and the renormalization group equation.

On the other hand, there is some freedom in dividing $C_0(\mu_R)$ into nonperturbative and perturbative pieces: $C_0(\mu_R) = C_0^{np}(\mu_R) + C_0^{p}(\mu_R)$, where $C_0^{np}(\mu_R) \sim 1/Q$ and $C_0^{p}(\mu_R) \sim Q^0$. This is simply a reorganization of the perturbative series. For instance, consider the following expansion of the amplitude in the theory without pions:

$$\begin{aligned} A &= \frac{4\pi}{M}\left[\frac{1}{-1/a + \frac{r_0}{2}p^2 + ... - ip}\right] = \frac{4\pi}{M}\left[\frac{1}{-1/a - \Delta + \Delta + \frac{r_0}{2}p^2 + ... - ip}\right] \\ &= \frac{-4\pi}{M}\left[\frac{1}{1/a + \Delta + ip} + \frac{\frac{r_0}{2}p^2 + \Delta}{(1/a + \Delta + ip)^2} + ...\right], \end{aligned} \tag{19}$$

where $\Delta \lesssim 1/a$. The series with $\Delta = 0$ and with $\Delta \neq 0$ will both reproduce effective range theory, but differ in the location of the pole that appears at each order in the perturbative expansion. In the ${}^3S_1$ channel, the pole of the physical amplitude is at $-ip = \sqrt{ME_d} = 45.7\,\mathrm{MeV}$, where E_d is the binding energy of the deuteron[1]. For comparison, $1/a = 36.3\,\mathrm{MeV}$ in this channel. For $\Delta = 0$, the pole that appears at each order in the perturbative expansion will be off by 30%. For some calculations, such as processes involving the deuteron [9], a better behaved perturbation series is obtained by choosing $1/a + \Delta = \sqrt{ME_d}$. If we want to reproduce the expansion in Eq. (19) in the theory without pions then part of $C_0(\mu_R)$ must be treated perturbatively.

In the theory with pions and a non-vanishing $C_0^p(\mu_R)$, the amplitude can be obtained from Eqs. (10–13) by substituting $C_0(\mu_R) \to C_0^{np}(\mu_R)$ and

$$A^{(-1)} \to A^{(-1)} - [A^{(-1)}]^2 \frac{C_0^p(\mu_R)}{C_0^{np}(\mu_R)^2}\,. \tag{20}$$

In the OS scheme, the renormalization group equation makes $C_0^p(\mu_R)/[C_0^{np}(\mu_R)]^2$ equal to a constant. Therefore, $C_0^p(\mu_R)$ can be simply absorbed into the definition of $D_2(\mu_R)$. Because of this, the value of $D_2(\mu_R)$ extracted from fits to NN scattering data may differ from the value of the renormalized coupling in the Lagrangian. In the PDS scheme, the renormalization group equation for $C_0^p(\mu_R)$ is

$$\mu_R \frac{\partial C_0^p(\mu_R)}{\partial \mu_R} = 2\,\frac{M\mu_R}{4\pi} C_0^{np}(\mu_R)\Big[C_0^p(\mu_R) + \frac{g^2}{2f^2}\Big]\,, \tag{21}$$

with solution

$$\frac{C_0^p(\mu_R)}{C_0^{np}(\mu_R)^2} + \frac{g^2}{2f^2}\frac{1}{C_0^{np}(\mu_R)^2} = \text{constant}\,. \tag{22}$$

Therefore, breaking $C_0(\mu_R)$ into perturbative and nonperturbative pieces results in a manifestly μ_R independent amplitude in PDS[2]. The constraint in Eq. (15) is now μ_R independent.

Integrating out the pion gives low-energy theorems for the coefficients v_i in the effective range expansion [10],

$$p\cot\delta(p) = -\frac{1}{a} + \frac{r_0}{2}\,p^2 + v_2\,p^4 + v_3\,p^6 + v_4\,p^8 + \ldots\,. \tag{23}$$

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[1]In fact, in the ${}^3S_1$ channel the fit value of $C_0(m_\pi)$ from Eq. (18) gives $\gamma = 47.3\,\mathrm{MeV}$.

[2]We would like to thank Mark Wise for pointing this out to us.

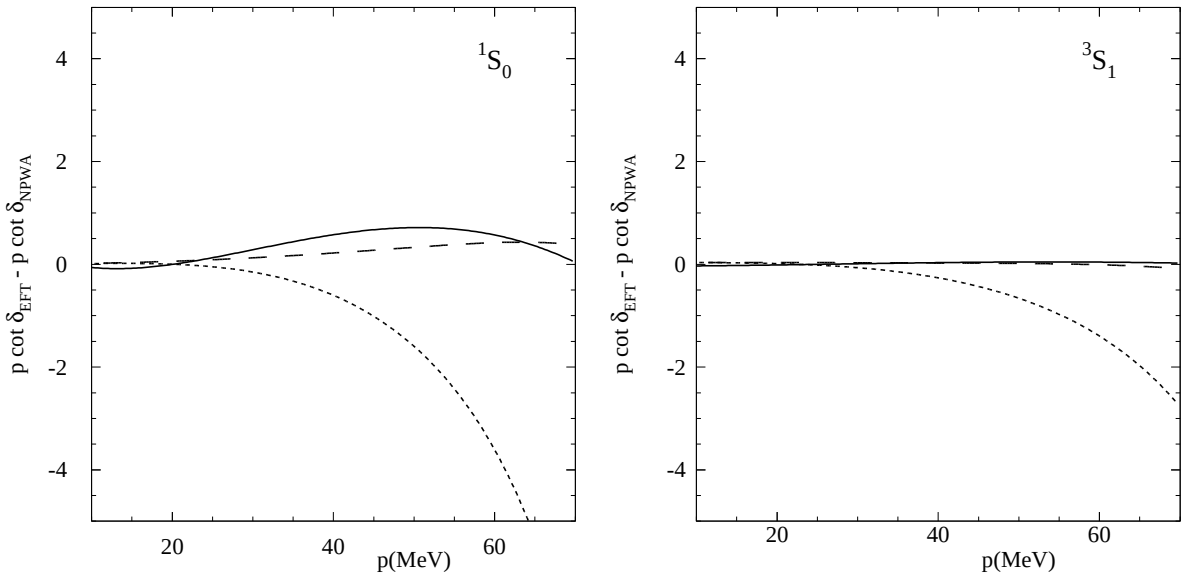

FIG. 3. The effective field theory and Nijmegen Partial Wave analysis [8] values of $p\cot\delta$ are compared. The solid lines use $p\cot(\delta^{(0)}+\delta^{(1)})$, the dashed lines use Eq. (23) with the v_i from Ref. [10], and the dotted lines use the values of v_i from the low-energy theorems.

The v_i can be predicted in terms of one parameter, $C_0^{np}(\mu_R)$, which is fixed in Ref. [10] by the condition $4\pi/[MC_0^{np}(\mu_R)]+\mu_R = 1/a$. Corrections to these predictions are expected to be $30-50\%$ due to higher order Q/Λ terms. The v_i extracted from the phase shift data [10,11] disagree with the low-energy theorems by factors of order 5. In Fig. 3, we see that the agreement of $p\cot(\delta^{(0)}+\delta^{(1)})$ (solid lines)[3] with the Nijmegen partial wave analysis is comparable to that of the effective range expansion with the v_i from the fits in Refs. [10,11] (dashed lines). Note that our fit is more accurate at low momentum than the global fit in Ref. [2]. However, keeping only the first five terms from the low-energy theorems (dotted lines) gives larger disagreement at 70 MeV. This is not surprising since the pion introduces a cut at $p = i\,m_\pi/2$, so the radius of convergence of the series expansion of $p\cot(\delta)$ in Eq (23) is $\simeq 70\,\mathrm{MeV}$. At $p = 70\,\mathrm{MeV}$, one expects large corrections from the next term in the series. However, the fit values of v_i give good agreement with the data even at 70 MeV. It is possible that uncertainty from higher order terms in the Taylor series has been absorbed into v_2, v_3, and v_4 in the process of performing the fits. For this reason, the uncertainty in the values of v_i that were found from fitting to the data may be considerable.

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[3]Note that when expanded in Q, $p\cot\delta = ip + 4\pi/[MA^{(-1)}] - 4\pi A^{(0)}/[M(A^{(-1)})^2] + \mathcal{O}(Q^3)$, which differs from $p\cot(\delta^{(0)}+\delta^{(1)})$ by terms of order Q^3. The latter expression is used since the parameters in Eq. (18) were fit using Eq. (17).

To get an idea of the error in v_2, we will specialize to the ${}^3S_1$ channel. The Nijmegen phase shift analysis [12] lists two data points for $p < 70\,\mathrm{MeV}$: $p = 21.67\,\mathrm{MeV}$ where $\delta^{({}^3S_1)} = 147.747 \pm 0.010\,^\circ$, and $p = 48.45\,\mathrm{MeV}$, where $\delta^{({}^3S_1)} = 118.178 \pm 0.021\,^\circ$. Using $a = 5.420 \pm 0.001\,\mathrm{fm}$ and $r_0 = 1.753 \pm 0.002\,\mathrm{fm}$ [11] in the effective range expansion and fitting to the lowest momentum data point, we find $v_2 = -0.50 \pm 0.52\,\mathrm{fm}^3$, where the error in a, r_0, and $p\cot\delta$ have been added in quadrature. This differs by one sigma from both the value predicted by the low-energy theorem, $v_2^{\mathrm{thm}} = -0.95\,\mathrm{fm}^3$, and the value from the fit, $v_2^{\mathrm{fit}} = 0.04\,\mathrm{fm}^3$. Since the range of the pure nucleon theory is $70\,\mathrm{MeV}$, there will also be a $\simeq 0.1\,\mathrm{fm}^3$ error in this extraction from v_3 and higher coefficients. This error was estimated by comparing the theoretical expression for $p\cot\delta$ with the first three terms in its series expansion. If we instead use the higher momentum point we find $v_2 = 0.03 \pm 0.04\,\mathrm{fm}^3$ with $\simeq 0.5\,\mathrm{fm}^3$ theoretical uncertainty. The uncertainty in these values of v_2 is too large to make a definitive test of the low-energy theorems.

In this paper, we have addressed the issue of μ_R sensitivity in perturbative treatments of the pion. Amplitudes are μ_R independent in the OS scheme, and in the PDS scheme, if part of $C_0(\mu_R)$ is treated perturbatively. Fits to NN scattering data were done which agree well at low momentum. Errors at high momentum are consistent with uncertainty from higher order terms if the range of the effective field theory is $\gtrsim 300\,\mathrm{MeV}$. We conclude that there is no obstruction to using perturbative pions for momenta $p > m_\pi$. In a future publication [7], we will describe the renormalization procedure and OS scheme in greater detail. The range of the theory with perturbative pions will also be investigated.

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